On the Determination of the Equations for the Perturbation of the Inclination and Node of an Orbit by the Method of Variation of Constants. By Robert Bryant, D.Sc., B.A.

The method adopted by Oppolzer and by Watson for determining the differential equations for the change in the position of the plane of the orbit seems to me less simple than the following.

Taking the sun's centre as origin, let three rectangular axes be chosen, whose positions will be more closely specified hereafter.

Let

x, y, z be the coordinates of the disturbed planet,

and

x', y', z' „ „ disturbing „

Then k being the Gaussian constant and p the semi-parameter of the orbit, we have

$$k\sqrt{p(1+m)}\cos i = xv_y - yv_x$$
$$k\sqrt{p(1+m)}\sin \Omega \sin i = yv_z - zv_y$$
$$-k\sqrt{p(1+m)}\cos \Omega \sin i = zv_x - xv_z$$

where v_x, v_y, v_x, are the velocities of the disturbed planet parallel to the coordinate axes.

By the differentiation of these equations and the introduction of suitable modifications we obtain the required results.

Now the quantities on the right-hand sides of the above equations are subject to two distinct kinds of variation: one arising from a change in the time only, such as occurs in the case of undisturbed motion; the other, that arising from the perturbation. Denote the former of these by the symbol $\frac{d_1}{dt}$, and the latter by $\frac{d_2}{dt}$, and let $\frac{d}{dt}$ denote the total variation, so that

$$\frac{d}{dt} = \frac{d_1}{dt} + \frac{d_2}{dt}.$$

Then putting for brevity

$$P = k\sqrt{p(1+m)},$$

we have

$$\frac{dP}{dt}\cos i - P\sin i\frac{di}{dt} = \frac{dx}{dt}v_y + x\frac{dv_y}{dt} - \frac{dy}{dt}v_x - y\frac{dv_x}{dt}.$$

$$\frac{dP}{dt}\sin \Omega \sin i + P\cos \Omega \sin i\frac{d\Omega}{dt} + P\sin \Omega \cos i\frac{di}{dt}$$
$$= \frac{dy}{dt}v_z + y\frac{dv_z}{dt} - \frac{dz}{dt}v_y - z\frac{dv_y}{dt}.$$

$$\frac{dP}{dt}\cos \Omega \sin i - P\sin \Omega \sin i\frac{d\Omega}{dt} + P\cos \Omega \cos i\frac{di}{dt}$$
$$= \frac{dz}{dt}v_x + z\frac{dv_x}{dt} - \frac{dx}{dt}v_z - x\frac{dv_z}{dt}.$$

The total differentials

$$\frac{dx}{dt}, \ \frac{dy}{dt}, \ \frac{dz}{dt}$$

are the velocities parallel to the coordinate axes.

Also

$$x\frac{dv_y}{at} - y\frac{dv_x}{dt} = x\left(\frac{d_1v_y}{dt} + \frac{d_2y}{dt}\right) - y\left(\frac{d_1v_x}{dt} + \frac{d_2v_x}{dt}\right)$$

$$= x\frac{d_1v_y}{dt} - y\frac{d_1v_x}{dt} + x\frac{d_2v_y}{dt} - y\frac{d_2v_x}{dt}.$$

Now at any instant the body is supposed to be describing an undisturbed orbit, in which case the area swept out by the radius vector is proportional to the first power of the time only, and its rate of change is therefore constant, as is also the projection upon the coordinate planes.

But

$$x\frac{d_1v_y}{dx} - y\frac{d_1v_x}{dt}$$

is proportional to the second differential with respect to the time of this area projected on the plane of xy, and therefore vanishes, as also do the corresponding quantities for the planes of yz and zx.

Also

$$\frac{d_2}{dt}(\text{velocity in any direction}) = \text{acceleration in that direction due to perturbation.}$$

Hence if X, Y, Z be the perturbative accelerations parallel to the respective coordinate axes, our equations are simplified to

$$\frac{dP}{dt}\cos i - P\sin i\frac{di}{dt} = xY - yX$$

$$\frac{dP}{dt}\sin \Omega \sin i + P\cos\Omega\sin i\frac{d\Omega}{dt} + P\sin\Omega\cos i\frac{di}{dt} = yZ - zY$$

$$\frac{dP}{dt}\cos\Omega\sin i - P\sin\Omega\sin i\frac{d\Omega}{dt} + P\cos\Omega\cos i\frac{di}{dt} = xZ - zX.$$

From the last two equations we deduce

$$\frac{dP}{dt}\sin i + P\cos i\frac{di}{dt} = (yZ - zY)\sin\Omega + (xZ - zX)\cos\Omega,$$

whence by aid of the first equation

$$P\frac{di}{dt} = (yZ - zY)\sin\Omega\cos i + (xZ - zX)\cos\Omega\cos i - (xY - yX)\sin i.$$

Also

$$P\sin i\frac{d\Omega}{dt} = (yZ - zY)\cos\Omega - (xZ - zX)\sin\Omega.$$

Hitherto we have left the positions of the rectangular coordinate axes quite arbitrary, except that their point of intersection is at the Sun's centre. Our equations will be still further sim-

plified by a suitable choice of axes. First, let the axis of x be directed towards the ascending node of the instantaneous orbit upon the ecliptic.

Then

$$P\frac{di}{dt} = (xZ - zX)\cos i - (xY - yX)\sin i,$$

and

$$P\sin i\frac{d\Omega}{dt} = yZ - zY.$$

Next, if we adopt the plane of the instantaneous orbit as the plane of xy, we have still more simply

$$P\frac{di}{dt} = xZ.$$

$$P\left[\sin i\frac{d\Omega}{dt}\right] = yZ.$$

Sin $i\, d\Omega$ is the projection of the perturbation of the node upon a great circle perpendicular to the plane of the instantaneous orbit.

There now only remains to determine the disturbing acceleration Z; X and Y now no longer appearing in our equations.

The fundamental equations of notion referred to axes originating in the Sun's centre are

$$\frac{d^2z}{dt^2} + k^2(1+m)\frac{z}{r^3} = k^2\Sigma m'\left(\frac{z'-z}{\rho^3}\right) - \frac{z'}{r'^3},$$

ρ being the distance between the disturbing and disturbed planets with two similar equations for x and y.

We will consider only one disturbing planet, the results for several planets merely requiring to be added together.

Then

$$Z = k^2m'\frac{z'-z}{\rho^3} - \frac{z'}{r^3}.$$

But as we have adopted the instantaneous plane of the orbit as the plane of xy, we have

$$z = 0.$$

Also if u be the argument of latitude,

$$x = r\cos u,\ y = r\sin u,$$

and substituting for P its value

$$k\sqrt{p(1+m)},$$

we have finally

$$\frac{di}{dt} = \frac{Kkm'}{\sqrt{p(1+m)}}z'r\cos u,$$

$$\sin i\frac{d\Omega}{dt} = \frac{Kkm'}{\sqrt{p(1+m)}}z'r\sin u,$$

where

$$K = \frac{1}{\rho^3} - \frac{1}{r^3}.$$

We might also have determined from the above equations the value of

$$\frac{dP}{dt},$$

and thence

$$\frac{dp}{dt},$$

and thus have found a relation between the variations of the major axis and the eccentricity. Thus our equations really determine the variations of three of the six elements of the orbit.

On the Variation of Latitude by the Greenwich Transit-Circle Observations. By S. C. Chandler.

The interesting communication of Messrs. Thackeray and Turner in the *Monthly Notices* for November 1892 was manifestly prepared before seeing the final law of the variation of latitude in No. 277 of the *Astronomical Journal.* I trust they will not deem it an encroachment on their province to point out that the general accordance of the observations with the provisional law of No. 267, which they have used, is still more satisfactory when the comparison is made with the definitive law. Without presuming to make a refined discussion, for which astronomers will look with interest to the continuation of the investigation by Messrs Thackeray and Turner, I find that the data of their Plate I. may be very satisfactorily represented by the expression

$$+0''\cdot09-0''\cdot141\cos(t-1885\text{ Apr. }23)\times0^\circ835-0''\cdot148\cos(\odot-300^\circ)\quad\text{(A)}$$

The law on p. 100, *Astronomical Journal*, No. 277, gives

$$-0''\cdot120\cos(t-1885\text{ Apr. }9)\times0^\circ\cdot835-0''\cdot110\cos(\odot-350^\circ).$$

The testimony of these observations is consequently entirely in harmony with the law in question, as will strikingly appear if the curve computed from equation (A) be entered upon Plate I. in the November *Monthly Notices.* Such a chart is given on Plate III.* The curve of equation (A) is represented by a dotted line; the observations by a zigzag continuous line, taking for the points of the latter the means of each successive three points, as given by Messrs. Thackeray and Turner, in order to eliminate, unobjectionably, the larger accidental errors.

These results tend still further to confirm the doubt expressed on p. 71, *Astronomical Journal*, No. 273, as to whether temperature-variation plays any significant part in the apparently anomalous results of this instrument, as has always been supposed. This matter has been the subject of a good deal of past investigation, but never with distinctly satisfactory conclusions; and in view of the new light which this freshly discovered annual

[* It was pointed out by Mr. Thackeray, but, unfortunately, not until after this plate had been printed off, that the theoretical curve is apparently one year in error. The portion for 1891 really belongs to 1890, and so on.—SECRETARIES.]